Stochastic averaging, large deviations and random transitions for the dynamics of 2D and geostrophic turbulent vortices

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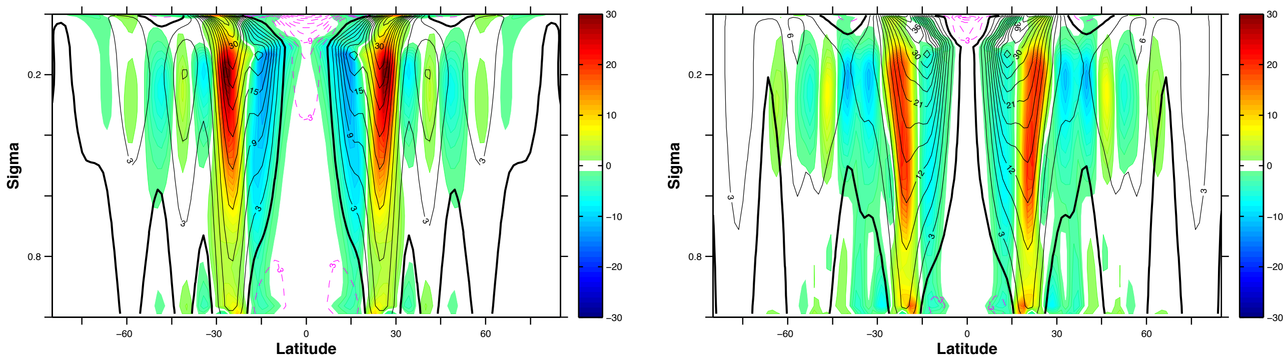
Abstract. Geophysical turbulent flows are characterized by their self-organisation into large scale coherent structures, in particular parallel jets. We will present a theory in order to describe the effective statistics and dynamics of these jets. We prove that this closure is exact in the limit of a time scale separation between the forcing and the inertial dynamics, which is rare in a turbulent flow. The equation obtained describes the attractors for the dynamics (alternating zonal jets), and the relaxation towards those attractors. At first order, these attractors are the same as the ones obtained from a quasi-Gaussian closure, already studied. Our work thus justifies this approximation and the corresponding asymptotic limit. We also present a new, very efficient algorithm to compute the terms appearing in this equation. The theory also goes beyond the quasi-Gaussian approximation, and indeed it can also describe the stationary distribution of the jets (fluctuations and large deviations).

Keywords: geostrophic turbulence; coherent structures; stochastic averaging

1. Introduction

The emergence of large-scale, long-lived, coherent structures is the main aspect of geophysical and astrophysical flows (Bouchet and Venaille 2012). The common pictures of Jupiter perfectly illustrate this fact: the surface flow is clearly organised into parallel, alternating zonal jets (parallel to the equator), with also a presence of giant and very stable vortices such as the Great Red Spot. Such large scale features are on one hand slowly dissipated, mainly due to a large-scale friction mechanism, and on the other hand maintained by the small-scale turbulence, through the Reynolds' stress. The main mechanism is thus a transfer of energy from the forcing scale (due to barotropic and baroclinic instabilities, or to small-scale convective activity) to the turbulent scales and until the scale of the jets. An important point in this phenomenology is the fact that the turbulent fluctuations are of very weak amplitude compared to the amplitude of the zonal jet, and that they evolve much faster. This means that the typical time scale of advection and shear of the fluctuations by the jet is much smaller than the typical time scale of formation or dissipation of the whole jet. This time scale separation is a very specific property of the geophysical large-scale structures, and it is a crucial element that will be stressed throughout the paper.

Numerical simulations of atmosphere flows can illustrate this phenomenon (figure 1, O'Gorman and Schneider 2007), and it is observed that the fully non-linear dynamics is not necessary to describe the formation of jets. The so-called *quasi-linear approximation* seems to be sufficient in order to reproduce quantitatively the dynamics of the jets.



(a) Full model. (b) Quasi-linear approximation.

Figure 1. Examples of direct numerical simulations of an atmospheric flow (primitive equations). The colors represent the intensity of the Reynolds' stress divergence (in units 10^{-6}m.s^{-2}), and the solid lines are the isolines of the zonally averaged flow (in units m.s^{-1}). We clearly see the formation of intense jets (25 m/s) in high altitudes and midlatitudes ($\sim 30°$). The result of the full simulations (left panel) is in very good agreement with the simulation of the quasi-linear dynamics (right panel), particularly when it comes to the averaged velocity (solid lines). Courtesy Farid Ait Chaalal.

In this turbulent context, the understanding of jet formation requires averaging out the effect of rapid turbulent degrees of freedom in order to describe the slow evolution of the jet structure. Such a task, an example of turbulent closure, is usually extremely hard to perfom for turbulent flows. We prove that it can be performed explicitly in this problem. It gives at leading order a quasi-Gaussian closure, which is naturally related to the quasi-linear dynamics presented above. The success of this approach strongly relies on the time scale separation mentionned earlier. We present in this paper the derivation of this closure (all the technical details of the derivation can be found in Bouchet *et al* 2013) and in section 3.2, we also present a new numerical algorithm used to simulate the quasi-linear dynamics.

Such linear or quasi-linear approaches have been commonly studied for decades in many theoretical discussions of geostrophic turbulence. Specifically for the problem of jet formation, such a quasi-linear approach is at the core of Stochastic Structural Stability Theory (S3T) first proposed in Farrell and Ioannou (2003) for quasi-geostrophic turbulence. More recently, an interpretation in terms of a second order closure (CE2) has also been given (Marston *et al* 2008, Tobias and Marston 2013). All these different forms of quasi-linear approximations have thoroughly been studied numerically, sometimes with stochastic forces and sometimes with deterministic ones (Marston *et al* 2008). Very interesting empirical studies (based on numerical simulations) have been performed recently in order to study the validity of this type of approximation (Marston *et al* 2008, Tobias and Marston 2013, Srinivasan and Young 2012), for the barotropic equations or for more complex dynamics. The S3T equations have also been used to study theoretically the transition from a turbulence without a coherent structure to a turbulence with zonal jets (Srinivasan and Young 2012).

The simplest model that leads to the formation of such jets is the one layer barotropic equations, on a beta-plane or over a topography, with stochastic forcing (Vallis 2006)

$$\partial_t q + \mathbf{v} \cdot \nabla q = -\lambda\omega - \nu_{n,d} \left(-\Delta\right)^n \omega + \sqrt{\sigma}\eta, \tag{1}$$

with the non-divergent velocity $\mathbf{v} = \mathbf{e}_z \times \nabla\psi$, the vorticity $\omega = \Delta\psi$ and the potential vorticity $q = \omega + \beta_d y$, where ψ is the stream function. We consider a doubly periodic domain $\mathcal{D} = [0, 2\pi L l_x) \times [0, 2\pi L)$ with aspect ratio l_x, but the results also apply for the dynamics in a channel, or in any bounded domain. λ is the Ekman friction coefficient, $\nu_{n,d}$ is a (hyper-)viscosity coefficient and β_d is the mean gradient of potential vorticity. η is a white in time gaussian random noise, with spatial correlation

$$\mathbf{E}\left[\eta(\mathbf{r}_1, t_1)\eta(\mathbf{r}_2, t_2)\right] = C(\mathbf{r}_1 - \mathbf{r}_2)\delta(t_1 - t_2),$$

which parametrizes the effect of baroclinic and barotropic instabilities. The correlation function C is assumed to be normalised such that σ represents the average energy injection rate, so that the average energy injection rate per unit of mass is $\epsilon = \sigma/4\pi^2 L^2 l_x$. In the regime we are interested in, the main energy dissipation mechanism is the linear friction, and the (hyper-)viscosity is negligible.

The evolution of the energy (averaged over noise realizations) E is given by

$$\frac{dE}{dt} = -2\lambda E + \sigma.$$

In a stationary state we have $E = E_{stat} = \sigma/2\lambda$, expressing the balance between forcing and dissipation. This expression gives the typical velocity of the coherent structure $U \sim \sqrt{E_{stat}}/L \sim \sqrt{\epsilon/2\lambda}$. Then the typical time scale of advection of a perturbation by the coherent structure is $\tau = L/U$. We can thus build a non-dimensional parameter α as the ratio of the advective time scale and the dissipative one $(1/\lambda)$ on which large scale structures typically evolve:

$$\alpha = \lambda\tau = L\sqrt{\frac{2\lambda^3}{\epsilon}}.$$

This parameter α is the small parameter by which the kinetic theory will be developed. Many works in literature (Vallis 2006 for instance) suggest that, when the beta effect and the non-linear effects are of the same order of magnitude, the largest relevant scale of the flow is given by the Rhines scale

$$L_R = (U/\beta_d)^{1/2} = \left(\epsilon/\beta_d^2\lambda\right)^{1/4}.$$

The time-scale of advection of a perturbation over this distance is $\tau_R = L_R/U$, then the ratio of the advective and dissipation time scales is given by $\alpha_R = \lambda\tau_R \propto \left(R_{\beta_d}\right)^{-5}$ where $R_{\beta_d} = \beta_d^{1/10}\epsilon^{1/20}\lambda^{-1/4}$ is the zonostrophy index introduced in Danilov and Gurarie (2004) or Galperin *et al* (2010). The regime $\alpha_R \ll 1$ thus coincides with the zonation regime $R_{\beta_d} \gg 1$, which is known to be the regime where the quasi-linear approximation is accurate (Tobias and Marston 2013).

In this regime, we have $\alpha_R < \alpha$. Then, in the regime $\alpha \ll 1$ considered in this work, we recover the zonation regime $\alpha_R \ll 1$.

The equations adimensionalized with the time scale τ and the length scale L read

$$\partial_t q + \mathbf{v}\cdot\nabla q = -\alpha\omega - \nu_n\left(-\Delta\right)^n\omega + \sqrt{2\alpha}\eta, \tag{2}$$

where $\nu_n = \nu_{n,d}\tau/L^{2n}$, $\beta = (L/L_R)^2$ with $\nu_n \ll \alpha \ll 1$. In the following, we will consider viscosity, $n = 1$, but the main results can be generalized to any type of hyper-viscosity.

2. Stochastic averaging

2.1. Rescaled dynamics

For the barotropic equations (2), the regime corresponding to the emergence of large-scale jets is given by $\alpha \ll 1$ (dynamical time scale of the jets much larger than the time scale of the turbulent fluctuations) and $\nu \ll \alpha$ (turbulent regime). This is the regime we consider in the following.

The large scale zonal jets are charecterized by either a zonal velocity field $\mathbf{v}_z(\mathbf{r}) = U(y)\mathbf{e}_x$ or its corresponding zonal potential vorticity $q_z(y) = -U'(y) + h(y)$. For reasons that will become clear in the following discussion (we will explain that this is a natural hypothesis and prove that it is self-consistent in the limit $\alpha \ll 1$), the non-zonal perturbation to this zonal velocity field is of order $\sqrt{\alpha}$. We then have the decomposition

$$q(\mathbf{r})=q_z(y)+\sqrt{\alpha}\omega_m(\mathbf{r}) \quad , \quad \mathbf{v}(\mathbf{r})=U(y)\mathbf{e}_x+\sqrt{\alpha}\mathbf{v}_m(\mathbf{r}) \tag{3}$$

where the zonal projection is defined by $\langle f\rangle(y)=\frac{1}{2\pi l_x}\int_0^{2\pi l_x}\mathrm{d}x\, f(\mathbf{r})$.

We now project the barotropic equation (2) into zonal and non-zonal part, assuming for simplicity that the random forcing does not act directly on the zonal degrees of freedom‡ ($\langle C\rangle=0$):

$$\frac{\partial q_z}{\partial t}=-\alpha\frac{\partial}{\partial y}\left\langle v_m^{(y)}\omega_m\right\rangle-\alpha\omega_z+\nu\frac{\partial^2\omega_z}{\partial y^2}, \tag{4}$$

$$\frac{\partial\omega_m}{\partial t}+L_U\left[\omega_m\right]=\sqrt{2}\eta-\sqrt{\alpha}\mathbf{v}_m.\nabla\omega_m+\sqrt{\alpha}\left\langle\mathbf{v}_m.\nabla\omega_m\right\rangle, \tag{5}$$

where

$$L_U[\omega_m]=U(y)\partial_x\omega_m+\partial_y q_z(y)\partial_x\psi_m$$

is the linearized dynamics operator around the zonal base flow U. We see that the zonal potential vorticity is coupled to the non-zonal one through the zonal average of the advection term $\frac{\partial}{\partial y}\left\langle v_m^{(y)}\omega_m\right\rangle$. If our rescaling of the equations is correct, we clearly see that the natural time scale for the evolution of the zonal flow is $1/\alpha$. By contrast, the natural time scale for the evolution of the non-zonal perturbation is one. These remarks show that in the limit $\alpha\ll 1$, we have a time scale separation between the slow zonal evolution and a rapid non-zonal evolution. Our aim is to use this remark in order to describe precisely the stochastic behavior of the Reynold stress in this limit (by integrating out the non-zonal turbulence), and to prove that our rescaling of the equations and this time scale separation hypothesis is a self-consistent hypothesis.

2.2. Adiabatic elimination of fast variables

We will use the remarks that we have a time scale separation between zonal and non-zonal degrees of freedom in order to average out the effect of the non-zonal turbulence. This amounts to treating the zonal degrees of freedom adiabatically. This kind of problems are described in the theoretical physics literature as adiabatic elimination of fast variables (Gardiner 1994) or may also be called stochastic averaging in the mathematics literature. Our aim is to perform the stochastic averaging of the barotropic flow equation and to find the equation that describes the slow evolution of zonal flows. In this stochastic problem, it is natural to work at the level of the probability density function (PDF) of the flow, $P[q]=P[q_z,\omega_m]$. Then, the dynamical equations (2) or (4) and (5) are equivalent to the so-called Fokker-Planck equation for P.

‡ This assmption is not necessary for the theory, it is just for convenience.

Complete Fokker-Planck equation The evolution equation for the PDF reads

$$\frac{\partial P}{\partial t} = \mathcal{L}_0 P + \sqrt{\alpha}\mathcal{L}_n P + \alpha \mathcal{L}_z P, \tag{6}$$

where

$$\mathcal{L}_0 P \equiv \int \mathrm{d}\mathbf{r}_1 \frac{\delta}{\delta \omega_m(\mathbf{r}_1)} \left[L_U \left[\omega_m\right](\mathbf{r}_1) P + \int \mathrm{d}\mathbf{r}_2\, C_m(\mathbf{r}_1 - \mathbf{r}_2) \frac{\delta P}{\delta \omega_m(\mathbf{r}_2)} \right] \tag{7}$$

is the Fokker-Planck operator that corresponds to the linearized dynamics close to the zonal flow U, forced by a Gaussian noise, white in time and with spatial correlations C. This Fokker-Planck operator acts on the non-zonal variables only and depends parametrically on U. This is in accordance with the fact that on time scales of order 1, the zonal flow does not evolve and only the non-zonal degrees of freedom evolve significantly. It should also be remarked that this term contains dissipation terms of order α and ν. These dissipation terms can be included in $\mathcal{L}_0$ because in the limit $\nu \ll \alpha \ll 1$, the non-zonal dynamics is dominated by the interaction with the mean flow, thanks to the so-called Orr mechanism (Orr 1907). This crucial point will be discussed in the following paragraph. At order $\sqrt{\alpha}$, the nonlinear part of the perturbation

$$\mathcal{L}_n P \equiv \int \mathrm{d}\mathbf{r}_1 \frac{\delta}{\delta \omega_m(\mathbf{r}_1)} \left[\left(\mathbf{v}_m . \nabla \omega_m(\mathbf{r}_1) - \left\langle \mathbf{v}_m . \nabla \omega_m(\mathbf{r}_1) \right\rangle \right) P \right] \tag{8}$$

describes the non-linear interactions between non-zonal degrees of freedom. At order α, the zonal part of the perturbation

$$\mathcal{L}_z P \equiv \int \mathrm{d}y_1 \frac{\delta}{\delta q_z(y_1)} \left[\left(\frac{\partial}{\partial y} \left\langle v_m^{(y)} \omega_m \right\rangle (y_1) + \omega_z(y_1) - \frac{\nu}{\alpha} \Delta \omega_z(y_1) \right) P \right. \tag{9}$$

$$\left. + \int \mathrm{d}y_2\, C_z(y_1 - y_2) \frac{\delta P}{\delta q_z(y_2)} \right] \tag{10}$$

contains the terms that describe the large-scale friction and the coupling between the zonal and non-zonal flow.

Stationary distribution of the fast variables The goal of our approach is to get an equation that describes only the zonal, slowly evolving part of the PDF, but taking into account the fact that the non-zonal degrees of freedom have rapidly relaxed to their stationary distribution. The first step is then to determine this stationary distribution of the non-zonal, fastly evolving degrees of freedom. This stationary distribution is given by the stationary state of (6), retaining only the first order term: $\mathcal{L}_0 P = 0$. For the special case of a determined zonal flow $P\left[q_z, \omega_m\right] = \delta\left(q_z - q_0\right) Q(\omega_m)$, $\mathcal{L}_0$ is the Fokker-Planck operator that corresponds to the dynamics of the non-zonal degrees of freedoms, for quasi-geostrophic equations linearized around the base flow with potential vorticity q_0,

$$\frac{\partial \omega_m}{\partial t} + L_U[\omega_m] = \sqrt{2}\eta. \tag{11}$$

It is a linear stochastic process (Orstein-Uhlenbeck process) with zero average value, so we know that its stationary distribution is a centered Gaussian, entirely determined by the variance of ω_m. The variance is the stationary value of the two-points correlation function of ω_m, $g(\mathbf{r}_1, \mathbf{r}_2, t) = \mathbf{E}\left[\omega_m(\mathbf{r}_1, t)\omega_m(\mathbf{r}_2, t)\right]$. The evolution of g is given by the so-called Lyapunov equation, which is obained by applying the Ito formula to (11)

$$\frac{\partial g}{\partial t} + \left(L_U^{(1)} + L_U^{(2)}\right) g = 2C. \tag{12}$$

($L_U^{(i)}$ means that the operator is applied to the i-th variable). We now understand that the asymptotic behaviour of this equation is a crucial point for the whole theory. It can be proved (Bouchet and Morita 2010, Bouchet *et al* 2013) that g has a well-defined limit (in the distributional sense) for $t \to \infty$, even in the absence of any dissipation mechanism ($\alpha = \nu = 0$). This may seem paradoxical as we deal with a linearized dynamics with a stochastic force and no dissipation mechanism. This is due to the Orr mechanism (Orr 1907, Bouchet and Morita 2010) (the effect of the shear through a non-normal linearized dynamics), that acts as an effective dissipation. The fact that (12) has a finite limit when $\alpha \to 0$ is the precise justification of the scaling (3), and it is thus the central point of the theory.

The average of an observable $A[q_z, \omega_m]$ over the stationary gaussian distribution is still a function of q_z, and it is an average over the non-zonal degrees of freedom, taking into account the fact that they have relaxed to their stationary distribution. In the following, we denote this average

$$\mathbf{E}_U\left[A\right] = \int \mathcal{D}[\omega_m]\, A[q_z, \omega_m] G\left[q_z, \omega_m\right], \tag{13}$$

the subscript U recalling that this quantity depends on the zonal flow. With this definition, the adiabatic reduction can be performed. The details of the computation, that follow Gardiner (1994), are reported in Bouchet *et al* (2013). Only the final result and its consequences are presented here.

Effective zonal Fokker-Planck equation The final Fokker-Planck equation for the slowly evolving part of the zonal jets PDF $R[q_z]$ reads:

$$\frac{\partial R}{\partial \tau} = \int \mathrm{d}y_1 \frac{\delta}{\delta q_z(y_1)} \left\{ \left[\frac{\partial F\left[U\right]}{\partial y_1} + \omega_z(y_1) - \frac{\nu}{\alpha}\frac{\partial^2 \omega_z}{\partial y_1^2} \right] R[q_z] \right. \tag{14}$$

$$\left. + \int \mathrm{d}y_2 \frac{\delta}{\delta q_z(y_2)} \left[C_R(y_1, y_2)\left[q_z\right] R\left[q_z\right] \right] \right\}, \tag{15}$$

which evolves over the time scale $\tau = \alpha t$, with the drift term

$$F\left[U\right] = \mathbf{E}_U\left[\left\langle v_m^{(y)} \omega_m \right\rangle\right](y_1) + \alpha F_1\left[U\right],$$

with F_1 a functional of q_z, and the diffusion coefficient $C_R(y_1, y_2)\left[q_z\right]$, that also depends on the zonal flow q_z.

This Fokker-Planck equation is equivalent to a non-linear stochastic partial differential equation for the potential vorticity q_z,

$$\frac{\partial q_z}{\partial \tau} = -\frac{\partial F}{\partial y}[U] - \omega_z(y_1) + \frac{\nu}{\alpha}\frac{\partial^2 \omega_z}{\partial y^2} + \zeta, \tag{16}$$

where ζ is white in time Gaussian noise with spatial correlation C_R. As C_R depends itself on the velocity field U, this is a non-linear noise. The main physical consequences and the numerical implementation of this equation are discussed in the following paragraphs.

3. Physical interpretation of the zonal Fokker-Planck equation

3.1. First order: quasi-linear dynamics

At first order in α, we obtain a deterministic evolution equation for q_z:

$$\frac{\partial q_z}{\partial t} = -\alpha \frac{\partial}{\partial y} \mathbf{E}_U \left[\left\langle v_m^{(y)} \omega_m \right\rangle \right] - \alpha \omega_z + \nu \Delta \omega_z. \tag{17}$$

To summarize, we found that at leading order in α, the zonal flow is forced by the average of the advection term due to the non-zonal fluctuations (Reynolds' stress), and that this quantity is computed from the *linearized* dynamics for the fluctuations. In other words, we could have applied the same stochastic reduction technique to the quasi-linear dynamics ((4) and (5) without non-linear terms), and we would have obtained at leading order the same deterministic equation (17). The system (17) and (12) is a quasi-Gaussian (or second-order) closure of the dynamics. Working directly at the level of the PDF, and using the tools of the stochastic reduction, we have been able to justify the closure of this problem. This quasi-Gaussian closure has been already studied in numerical works (SSST in Farrell and Ioannou (2003) and CE2 in Marston *et al* (2008)) and analytical works (Srinivasan and Young 2012), and is known to give very good results.

Using again the results about the Orr mechanism (Orr 1907, Bouchet and Morita 2010, Bouchet *et al* 2013), some important facts about equation (17) can be proved. First, we can make sure that the Reynolds' stress is well-behaved, even in the inertial limit $\alpha, \nu \to 0$, so that the zonal flow equation (17) is always well-defined. We can also show that the energy in the non-zonal degrees of freedom is of order α. As a consequence, a vanishing amount of energy is dissipated in the fluctuations and almost all the energy injected by the stochastic forcing goes to the zonal degrees flow. Moreover, the dynamics defined by (17) and (12) are much simpler to solve numerically than the full non-linear dynamics (2). A very efficient algorithm used to compute the forcing term $-\alpha \frac{\partial}{\partial y} \mathbf{E}_U \left[\left\langle v_m^{(y)} \omega_m \right\rangle \right]$ is presented in the next section 3.2.

3.2. Numerical evaluation of the Reynolds' stress

In this section we present an efficient algorithm used to compute the forcing term $\mathbf{E}_U \left[\left\langle v_m^{(y)} \omega_m \right\rangle \right]$ appearing in the first order equation for the slow evolution of zonal jets (17). We recall that this quantity is the limit for infinite time of the statistical and zonal average of $v_m^{(y)} \omega_m$, where ω_m evolves according to the linearized stochastic dynamics (11), with the base flow U held fixed. Equivalently, this quantity can be computed as a linear transform of the infinite-time limit of the solution of the Lyapunov equation (12). In the limit $\nu \ll \alpha \ll 1$, the classical numerical resolution of (12) obtained discretising the operators $L_U^{(1)}$ and $L_U^{(2)}$ is a very difficult task. Indeed, the discretisation step should be taken smaller and smaller as ν goes to 0. On the contrary, with the algorithm presented below, we can directly take the limit $\nu = 0$.

The forcing correlation function can be expanded as $C_m(\mathbf{r}) = \sum_{k>0,l} c_{k,l} \cos(kx + ly)$, with $c_{k,l} \geq 0$. Then we can decompose the vorticity field as $\omega_m(\mathbf{r},t) = \sum_{k,l=-\infty}^{+\infty} \sqrt{c_{k,l}}\, \omega_{k,l}(y,t)\mathrm{e}^{ikx}$ where the coefficients satisfy

$$\partial_t \omega_{k,l} + L_{U,k}[\omega_{k,l}] = \mathrm{e}^{ily}\eta_{k,l}(t), \tag{18}$$

with

$$L_{U,k} = ikU - ik(U'' - \beta)\Delta_k^{-1} + \alpha, \tag{19}$$

the $k-$th component of the linear operator L_U. The white noises satisfy $\eta^*_{k,l} = \eta_{-k,-l}$ and $\mathbf{E}[\eta_{k_1,l_1}(t_1)\eta^*_{k_2,l_2}(t_2)] = \delta_{k_1,k_2}\delta_{l_1,l_2}\delta(t_1 - t_2)$, and $c_{k,l}$ is defined for $k < 0$ by $c_{k,l} = c^*_{-k,-l} = c_{-k,-l}$. With this decomposition, the Reynolds' stress divergence reads

$$\mathbf{E}_U\left[\left\langle v_m^{(y)}\omega_m\right\rangle\right] = \sum_{k,l=-\infty}^{+\infty} -ikc_{k,l}\, h_{k,l}(y) = \sum_{k>0,l} 2kc_{k,l}\, \mathrm{Im}[h_{k,l}(y)], \tag{20}$$

with the vorticity-stream function correlation function $h_{k,l} = \mathbf{E}_U[\omega_{k,l}\psi^*_{k,l}]$. In (Bouchet *et al* 2013), we presented a numerical scheme for the computation of $h_{k,l}$ based on an integral equation. This algorithm turns out to be efficient only in very specific cases. Here we present a much more efficient way to compute $h_{k,l}$, based on the computation of the resolvent of the operator $L_{U,k}$. From the formal solution of the Ornstein-Uhlenbeck process (18), we have

$$h_{k,l}(y) = \int_{-\infty}^{\infty} \mathrm{d}t\, \tilde{\omega}_{k,l}(y,t)\tilde{\psi}^*_{k,l}(y,t), \tag{21}$$

where $\tilde{\omega}_{k,l}(y,t)$ is the vorticity field obeying the deterministic initial value problem

$$\begin{aligned} &\forall t > 0, \quad \partial_t\tilde{\omega}_{k,l}(y,t) + L_{U,k}\left[\tilde{\omega}_{k,l}\right](y,t) = 0, \\ &\tilde{\omega}_{k,l}(y,0) = \mathrm{e}^{ily}, \\ &\forall t < 0, \quad \tilde{\omega}_{k,l}(y,t) = 0, \end{aligned}$$

and $\tilde{\psi}_{k,l}(y,t)$ is the associated stream function.

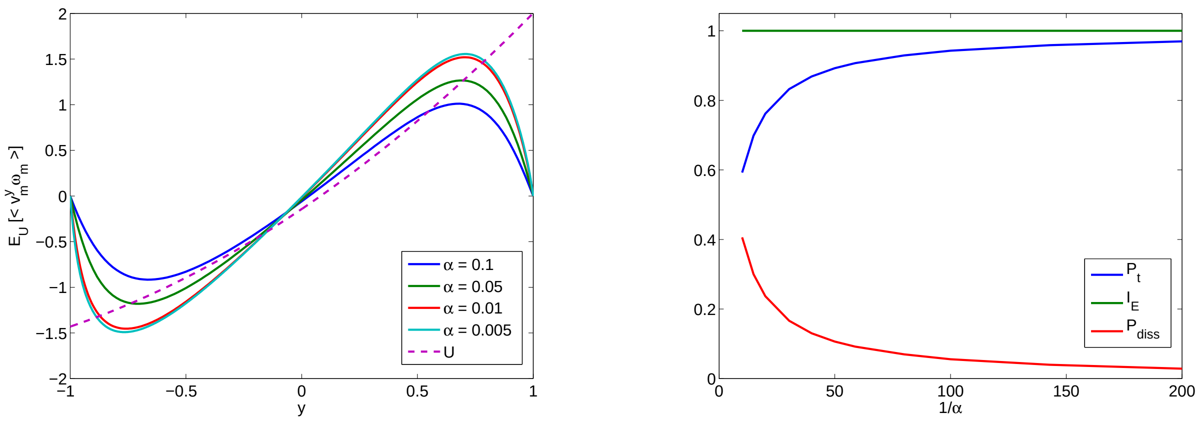
We move now to the frequency domain. The Laplace transform of the vorticity is denoted with

$$\hat{\omega}_{k,l}(y,c) = \int_0^{\infty} \mathrm{d}t\, \tilde{\omega}_{k,l}(y,t)\mathrm{e}^{ikct} = \int_{-\infty}^{\infty} \mathrm{d}t\, \tilde{\omega}_{k,l}(y,t)\mathrm{e}^{ikct}. \tag{22}$$

For all $\alpha > 0$, the deterministic vorticity field $\tilde{\omega}_{k,l}$ decays to 0 for $t \to \infty$. Then, the Laplace transform defined by (22) for real values of c coincides with the Fourier transform with respect to t. As a consequence, it can be simply inverted as

$$\tilde{\omega}_{k,l}(y,t) = \frac{|k|}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}c\, \hat{\omega}_{k,l}(y,c)\mathrm{e}^{-ikct}. \tag{23}$$

We stress that this property is valid only for decaying fields $\tilde{\omega}_{k,l}$, and thus only when $\alpha \neq 0$.



(a) The Reynolds' stress divergence and the zonal flow (dashed line).

(b) Stationary energy balance for the fluctuations.

Figure 2. Numerical results in the case of a parabolic base zonal flow $U(y) = A(y+2)^2 + U_0$ in a channel geometry, with a forcing at the scale $k_x = 1$, $k_y = 1$ and different values of the friction coefficient α, and $\nu = 0$. We see that in the inertial limit $\alpha \to 0$, the Reynolds' stress converges to a well-defined function, and that all the energy injected in the fluctuations is transferred to the zonal flow (blue line), while the energy dissipated in the fluctuations vanishes (red line). This constitutes a verification of the self-consistency of the theory, and relies on the non trivial Orr mechanism.

It is also useful to define the Laplace transform of the stream function $\hat{\psi}_{k,l}$; this quantity is usually referred in literature as the resolvent of the operator $L_{U,k}$. It is related to $\hat{\omega}_{k,l}$ through $\hat{\omega}_{k,l}(y,c) = \left(\frac{d^2}{dy^2} - k^2\right)\hat{\psi}_{k,l}(y,c)$ and is the solution of the linear ordinary differential equation

$$\left(\frac{d^2}{dy^2} - k^2\right)\hat{\psi}_{k,l}(y,c) - \frac{U''(y) - \beta}{U(y) - c - i\frac{\alpha}{k}}\hat{\psi}_{k,l}(y,c) = \frac{\mathrm{e}^{ily}}{ik\left(U(y) - c - i\frac{\alpha}{k}\right)}. \quad (24)$$

Using (23) and (24) in (21), and performing the integration over t, we get

$$h_{k,l}(y) = \frac{|k|}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}c\, \frac{(U''(y) - \beta)\hat{\psi}_{k,l}(y,c) + \mathrm{e}^{ily}/ik}{U(y) - c - i\frac{\alpha}{k}}\,\hat{\psi}^*_{k,l}(y,c). \quad (25)$$

The numerical computation of the resolvant $\hat{\psi}_{k,l}$ is a very easy task, an algorithm is detailed in Bouchet and Morita (2010). For a given value of c, computing $\hat{\psi}_{k,l}(y,c)$ takes a few seconds on a laptop. Then, the computation of the integral (25) is a matter of a few minutes, with a very good accuracy (of the order of 10^{-3}). The fact that we are able to compute the infinite-time limit of a statistical average, in the limit of no viscosity, so fast and with such accuracy, is an important result in itself. An example of application of this method is given in figure 2. Using (25) for smaller and smaller values of α allows to check direclty the theoretical statements of the previous paragraph 3.1: the Reynolds' stress divergence $\mathbf{E}_U\left[\left\langle v_m^{(y)}\omega_m\right\rangle\right]$ converges to a well-defined function in the limit $\alpha \to 0$, and all the energy is transferred from the forcing to the zonal flow in this limit.

3.3. Next order: corrections and multistability

From the full Fokker-Planck equation (6), we expect the non-linear operator $\mathcal{L}_n$ to produce terms of order $\alpha^{1/2}$ and $\alpha^{3/2}$ in the zonal Fokker-Planck equation (15). The detailed computation shows that these terms exactly vanish (Bouchet *et al* 2013). As a consequence, we have proved that the quasi-Gaussian closure (17,12) is correct in the limit $\alpha \ll 1$, with correction only at order α^2.

We then have a correction F_1 to the drift $F[U]$ due to the non-linear interactions. At this order, the quasilinear dynamics and non-linear dynamics differ. We also see the appearance of the noise term, which has a qualitatively different effect than the drift term. For instance if one is interested in large deviations from the most probable states, correction of order α to F_0 will still be vanishingly small, whereas the effect of the noise will be essential. This issue is important for the description of the bistability of zonal jets and phase transitions.

4. Conclusion

We have shown that it is possible to average out the effect of turbulence in the problem of the jet formation in the barotropic model, when there is a time scale separation between the evolution of the jets and the turbulent dynamics. Instead of following the classical route, based on an arbitrary closure in the Reynolds' hierarchy of equations, we performed this closure working directly at the level of the probability distribution function of the vorticity field (Bouchet *et al* 2013). The main aspects of the equation we obtain are the following: at first order, it describes the quasi-Gaussian closure, and thus justifies theoretically the previous studies on the subject. We presented here a new and very efficient scheme to compute numerically the quantities appearing in this equation. Then, it predicts the corrections to this approximation, and allows the description of the bistability of the jets. This last point is the subject of ongoing research.

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